Breaking-rate minimum predicts the collapse point of overloaded materials

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As a model of composite materials, we choose a bundle of fibers with stochastically distributed breaking thresholds for the individual fibers. The fibers are assumed to share the load equally, and to obey Hookean elasticity right up to the breaking point. We study the evolution of the fiber breaking rate at a constant load in excess of the critical load. The analysis shows that the breaking rate reaches a minimum when the system is half-way from its complete collapse.

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I. INTRODUCTION

Bundles of fibers, with statistical distributed thresholds for the breakdown of individual fibers, present interesting models of failures in materials. They have simple geometry and clear-cut rules for how stress caused by a failed element is redistributed on undamaged fibers. Since these models can be analyzed to an extent that is not possible for more complex materials, they have been much studied (For reviews, see [1–5]). The statistical distribution of the *size* of avalanches in fiber bundles is well studied [6–8], and the failure dynamics under constant load has been formulated [9] through recursion relations which in turn explore the phase transitions and associated critical behavior in these models.

In this paper we present a way to predict when an overloaded bundle collapses, by monitoring the fiber breaking rate. We focus on the equal-load-sharing models, in which the load previously carried by a failed fiber is shared equally by all the remaining intact fibers [10–13]. We consider a bundle consisting of a large number N of elastic fibers, clamped at both ends (Fig. 1). The fibers obey Hooke's law with force constant set to unity for simplicity. Each fiber i is associated with a breakdown threshold x_i for its elongation. When the length exceeds x_i the fiber breaks immediately, and does not contribute to the strength of the bundle thereafter. The individual thresholds x_i are assumed to be independent random variables with the same cumulative distribution function $P(x)$ and a corresponding density function $p(x)$ as follows:

$$\text{Prob}(x_i < x) = P(x) = \int_0^x p(y)dy. \tag{1}$$

If an external load F is applied to a fiber bundle, the resulting failure events can be seen as a sequential process [9]. In the first step all fibers that cannot withstand the applied load break. Then the stress is redistributed on the surviving fibers, which compels further fibers to fail, etc. This iterative process continues until all fibers fail, or an equilibrium situation with a nonzero bundle strength is reached. Since the number of fibers is finite, the number of steps, t_f, in this sequential process is *finite*.

At a force (or elongation) x per surviving fiber the total force on the bundle is x times the number of *intact* fibers. The expected or average force at this stage is therefore

$$F(x) = Nx[1 - P(x)]. \tag{2}$$

The maximum F_c of $F(x)$ corresponds to the value x_c for which dF/dx vanishes. Thus

$$1 - P(x_c) - x_c p(x_c) = 0. \tag{3}$$

We characterize the state of the bundle as *precritical* or *postcritical* depending upon the stress value $\sigma = F/N$ relative to the critical stress

$$\sigma_c = F_c/N. \tag{4}$$

We study the stepwise failure process in the bundle, when a fixed external load $F = N\sigma$ is applied. Let N_t be the number of intact fibers at step no. t, with $N_0 = N$. We want to determine how N_t decreases until the degradation process stops. With N_t intact fibers, an expected number

$$[NP(N\sigma/N_t)] \tag{5}$$

of fibers will have thresholds that cannot withstand the load, and consequently these fibers break immediately. Here $[X]$ denotes the largest integer not exceeding X. The number of intact fibers in the next step is therefore

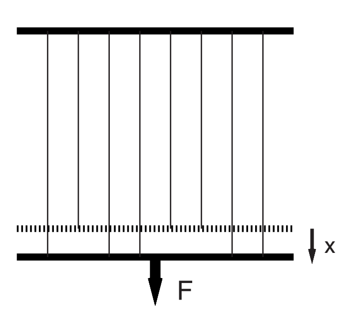

FIG. 1. The fiber bundle model.

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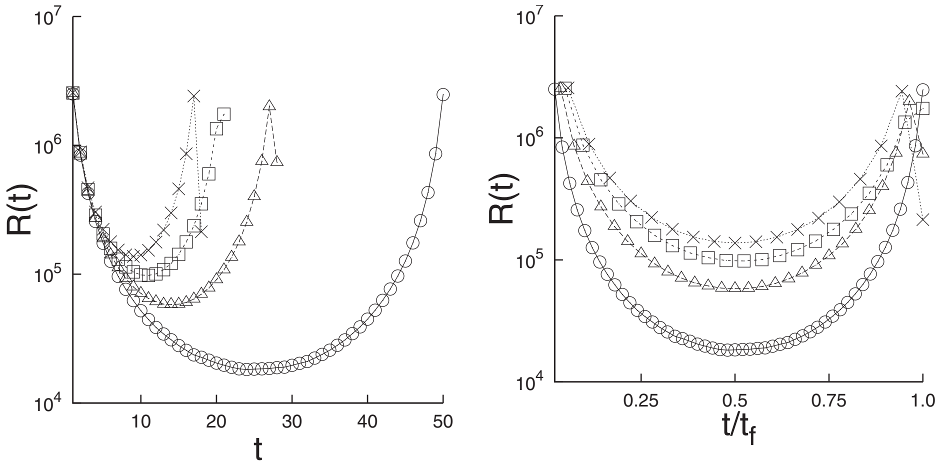

FIG. 2. The breaking rate $R(t)$ vs step t (left-hand plot) and vs the rescaled step variable t/t_f (right-hand plot) for the uniform threshold distribution for a bundle of $N=10^7$ fibers. Different symbols are used for different excess stress levels $\sigma-\sigma_c$: 0.001 (circles), 0.003 (triangles), 0.005 (squares), and 0.007 (crosses).

$$N_{t+1}=N-[NP(N\sigma/N_t)]. \tag{6}$$

Since N is a large number, the effect of the largest integer construction is negligible. Consequently iteration (6) is essentially [9]

$$n_{t+1}=1-P(\sigma/n_t) \tag{7}$$

in terms of the ratio

$$n_t=\frac{N_t}{N}. \tag{8}$$

II. RELATION BETWEEN MINIMUM BREAKING RATE AND COMPLETE COLLAPSE

We will now demonstrate, for three different threshold distributions, that there is a relation between the minimum of the breaking rate $R(t)=-dn_t/dt$ (treating t as continuous) and the moment t_f when the complete fiber bundle collapses.

A. Uniform distribution

We consider the uniform distribution, $P(x)=x$ for $0\le x\le 1$, and assume that the load is postcritical: $\sigma=\frac{1}{4}+\epsilon$, with $\epsilon>0$. Simulations show that the breaking rate has a minimum at some value $t_0(\epsilon)$, and that for varying ϵ the minima all occur at a value close to $\frac{1}{2}$ when plotted as function of the scaled variable t/t_f (Fig. 2).

This can be shown analytically. Iteration (7) takes in this case the form

$$n_{t+1}=1-\left(\frac{1}{4}+\epsilon\right)\frac{1}{n_t}. \tag{9}$$

By direct insertion one verifies that

$$n_t=\frac{1}{2}-\sqrt{\epsilon}\tan(At-B), \tag{10}$$

where

$$A=\tan^{-1}(2\sqrt{\epsilon}) \quad \text{and} \quad B=\tan^{-1}(1/2\sqrt{\epsilon}), \tag{11}$$

is the solution to Eq. (9) satisfying the initial condition $n_0=1$. From Eq. (10) follows the breaking rate

$$R(t)=-\frac{dn_t}{dt}=\sqrt{\epsilon}A\cos^{-2}(At-B). \tag{12}$$

$R(t)$ has a minimum when

$$0=\frac{dR}{dt}\propto \sin(2At-2B), \tag{13}$$

which corresponds to

$$t_0=\frac{B}{A}. \tag{14}$$

When criticality is approached, i.e., when $\epsilon\to 0$, we have $A\to 0$, and thus $t_0\to\infty$, as expected.

We see from Eq. (10) that $n_t=0$ for

$$t_f=[B+\tan^{-1}(1/2\sqrt{\epsilon})]/A=2B/A. \tag{15}$$

This is an excellent approximation to the integer value at which the fiber bundle collapses completely.

Thus with very good approximation we have the simple connection

$$t_f=2t_0. \tag{16}$$

When the breaking rate starts increasing we are halfway to complete collapse.

B. Displaced uniform distribution

Consider a uniform distribution on the interval $(x_l,1)$ as follows:

$$p(x)=\begin{Bmatrix}\frac{1}{1-x_l}, & x_l\le x\le 1\\ 0, & \text{otherwise}\end{Bmatrix}. \tag{17}$$

Thus

$$P(x)=\begin{Bmatrix}0, & x<x_l\\ \frac{x-x_l}{1-x_l}, & x_l\le x\le 1\end{Bmatrix}. \tag{18}$$

Simulations of the breaking rate gives qualitatively the same behavior as for the uniform distribution (Fig. 3). For this distribution Eq. (2) gives

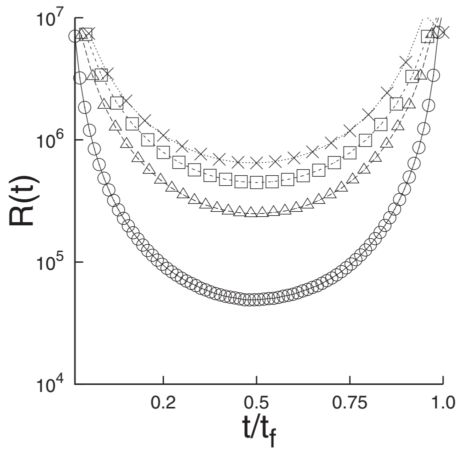

FIG. 3. The breaking rate $R(t)$ vs the rescaled step variable t/t_f for displaced uniform threshold distribution (17). Here $x_l=0.2$ and $N=5\times10^7$. Different symbols are used for different excess stress levels $\sigma-\sigma_c$: 0.001 (circles), 0.003 (triangles), 0.005 (squares) and 0.007 (crosses).

$$\sigma = x[1-P(x)] = \frac{x(1-x)}{1-x_l}, \tag{19}$$

with a maximum

$$\sigma_c = \frac{1}{4(1-x_l)}. \tag{20}$$

at $x=x_c=1/2$.

Iteration (7) takes now the form

$$n_{t+1} = \frac{1-\sigma/n_t}{1-x_l}. \tag{21}$$

This can be cast in a familiar form. Introduce

$$y_t = n_t(1-x_l) \tag{22}$$

in Eq. (21) to obtain the iteration for y_t as follows:

$$y_{t+1} = 1-\sigma(1-x_l)\frac{1}{y_t}. \tag{23}$$

By Eq. (20) the critical value of $\sigma(1-x_l)$ is 1/4, so we may write

$$\sigma(1-x_l) = \frac{1}{4}+\epsilon, \tag{24}$$

where again ϵ is assumed to be small and positive. Then we are back to the same iteration [Eq. (9)] as for the usual uniform distribution

$$y_t = \frac{1}{2} - \sqrt{\epsilon}\tan\left[\tan^{-1}\left(\frac{\frac{1}{2}-y_0}{\sqrt{\epsilon}}\right) + t\tan^{-1}(2\sqrt{\epsilon})\right] \tag{25}$$

or, since by Eq. (22) and $n_0=1$ we have $y_0=n_0(1-x_l)=1-x_l$

$$y_t = \frac{1}{2} - \sqrt{\epsilon}\tan[-\tan^{-1}((1/2-x_l)/\sqrt{\epsilon}) + t\tan^{-1}(2\sqrt{\epsilon})]. \tag{26}$$

For simplicity write this as

$$y_t = \frac{1}{2} - \sqrt{\epsilon}\tan(at-b), \tag{27}$$

with

$$a=\tan^{-1}(2\sqrt{\epsilon}) \quad \text{and} \quad b=\tan^{-1}((1/2-x_l)/\sqrt{\epsilon}). \tag{28}$$

The breaking rate (treating t as continuous) is

$$R(t) = -\frac{dn_t}{dt} = -\frac{1}{1-x_l}\frac{dy_t}{dt} = \frac{\sqrt{\epsilon}}{1-x_l}\cos^{-2}(at-b). \tag{29}$$

The minimum breaking rate occurs when $dR/dt \propto \sin(2at-2b)=0$, i.e., at $t_0=b/a$. For small ϵ we use the identity

$$\tan^{-1}(1/\eta) = \pi/2 - \tan^{-1}(\eta), \tag{30}$$

and obtain approximately for small ϵ

$$a \approx 2\sqrt{\epsilon} \text{ and } b \approx \pi/2 - \sqrt{\epsilon}/(1/2-x_l). \tag{31}$$

Using this, we obtain to leading order

$$t_0 \simeq \frac{\pi}{4\sqrt{\epsilon}}. \tag{32}$$

A good approximation to the collapse point t_f is obtained by selecting the t for which n_t or y_t vanishes. From Eq. (27) we see that this occurs for a t_f given by

$$\frac{1}{2} - \sqrt{\epsilon}\tan(at_f-b) = 0, \tag{33}$$

i.e.,

$$t_f = [b+\tan^{-1}(1/2\sqrt{\epsilon})]/a. \tag{34}$$

Again, by using Eq. (30) we have for small ϵ

$$t_f = \frac{\pi/2 - \sqrt{\epsilon}/(1/2-x_l) + \pi/2 - 2\sqrt{\epsilon}}{2\sqrt{\epsilon}}, \tag{35}$$

$$= \frac{\pi}{2\sqrt{\epsilon}}[1+\mathcal{O}(\sqrt{\epsilon})]. \tag{36}$$

Comparing the results for t_0 and t_f we have once more

$$t_f/t_0 = 2 \tag{37}$$

to leading order.

C. Weibull distribution

Let us finally consider a completely different threshold distribution, a Weibull distribution of index 5, $P(x)=1-e^{-x^5}$. Simulations reveal that the breaking rate has a similar behavior as in the two cases considered above (Fig. 4).

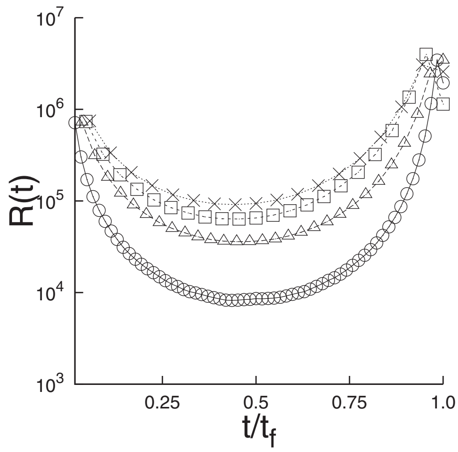

FIG. 4. The breaking rate $R(t)$ vs. the rescaled step variable t/t_f for a bundle of $N=10^7$ fibers having a Weibull threshold distribution. Different symbols are used for different excess stress levels $\sigma-\sigma_c$: 0.001 (circles), 0.003 (triangles), 0.005 (squares) and 0.007 (crosses).

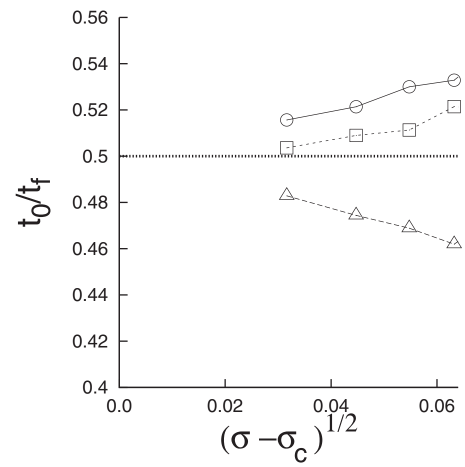

FIG. 5. Simulation results for the ratio t_0/t_f vs. $(\sigma-\sigma_c)^{-1/2}$ for the uniform distribution (circles), the displaced uniform distribution with $x_l=0.2$ (squares) and for the Weibull distribution (triangles). The graphs are based on 1000 samples with $N=10^7$ fibers.

This case is more complicated, but the analytical ground work has already been done in [14]. Equation (29) in [14] shows that for small ϵ the iteration is of the form

$$n_t = n_c - b\sqrt{\epsilon/C}\,\tan(t\sqrt{C\epsilon}-c). \tag{38}$$

Here $n_c=e^{-1/5}$, $C=\frac{5}{2}(5e)^{1/5}$, $b=5^{1/5}$ and the constant c is determined by the initial condition $n_0=1$

$$c = \tan^{-1}[(1-n_c)b^{-1}\sqrt{C/\epsilon}]. \tag{39}$$

From Eq. (38), the breaking rate equals

$$R(t) = -\frac{dn_t}{dt} \propto \cos^{-2}(t\sqrt{C\epsilon}-c). \tag{40}$$

The breaking rate is a minimum when the cosine takes its maximum value 1. This is the case when

$$t_0 = \frac{c}{\sqrt{C\epsilon}} = (C\epsilon)^{-1/2}\tan^{-1}[(1-n_c)b^{-1}\sqrt{C/\epsilon}]. \tag{41}$$

The inverse tangent is close to $\pi/2$ when ϵ is very small. Hence, for small overloads, we have in excellent approximation

$$t_0 = \frac{\pi}{2\sqrt{C\epsilon}}. \tag{42}$$

The collapse point t_f is already evaluated in [14], with the result

$$t_f \simeq \frac{\pi}{\sqrt{C\epsilon}} \tag{43}$$

for small ϵ [Eq. (33) in [14]].

Comparison between Eqs. (43) and (42) gives

$$t_f \simeq 2t_0, \tag{44}$$

as for the two previous threshold distributions considered.

III. COMMENTS

We have shown that the complete collapse of fiber bundles occurs at $t_f=2t_0$, where t_0 denotes the number of steps of the breaking process at which the fiber breaking rate has a minimum. The results are derived for very small overloads ϵ. For larger overloads the ratio t_0/t_f will not be exactly 0.5, as illustrated in Fig. 5, but nevertheless of the order of 0.5.

Another interesting observation is that at $t=t_0$ the number of unbroken fibers in the bundle $n(t_0)$ attains the critical value n_c. This can be derived analytically by putting the value of t_0 in expressions (10), (27), and (37), respectively, for the uniform, the displaced uniform, and the Weibull distribution. The numerical simulations (Fig. 6) strongly support this result.

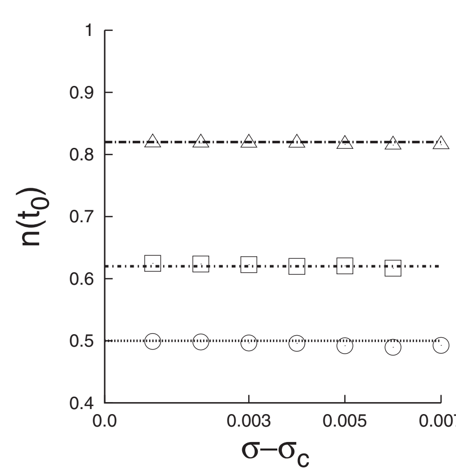

FIG. 6. Simulation results for $n(t_0)$ vs. $(\sigma-\sigma_c)$ for the uniform distribution (circles), the displaced uniform distribution with $x_l=0.2$ (squares) and for the Weibull distribution (triangles). The graphs are based on 1000 samples with $N=10^7$ fibers. The straight lines represent the critical value n_c for these three distributions.

IV. SUMMARY

In summary, we have considered slightly overloaded fiber bundles, and investigated how the fiber breaking rate progresses. It has a minimum after a number of steps t_0 of the degradation process, and we have demonstrated that the total bundle collapse occurs near $2t_0$. The demonstration has been performed for three different distributions of fiber thresholds, but the result is doubtlessly universal for equal-load-sharing models. Thus the fact that the breaking rate has a minimum predicts not only *that* a global failure will occur, but also estimates *when* it will occur. It would be interesting to see if similar predictions can be made for other models, like the local-load-sharing model.

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